Enrollment No: ______________________ Exam Seat No:______________________

C.U.SHAH UNIVERSITY

Summer Examination-2016

Subject Name : Algebra-I

Subject Code : 5SC02MTC5 **Branch: M.Sc.(Mathematics)**

Semester : 2 **Date : 13/05/2016** **Time : 10:30 To 01:30** **Marks : 70**

Instructions:

(1) Use of Programmable calculator and any other electronic instrument is prohibited.
(2) Instructions written on main answer book are strictly to be obeyed.
(3) Draw neat diagrams and figures (if necessary) at right places.
(4) Assume suitable data if needed.

SECTION – I

Q-1 **Attempt the Following questions** **(07)**

a. State all the unit elements of $(\mathbb{Z}[i];\ +;\ \cdot)$. **(02)**
b. Define: prime element **(02)**
c. Show that the polynomial $x^2 + 1$ is irreducible over $\mathbb{R}$. Is it reducible over $\mathbb{C}$? **(02)**
d. Show that the polynomial $x^2 + 2$ has one zero in $\mathbb{Z}_2$. **(01)**

Q-2 **Attempt all questions** **(14)**

a. Show that the ring $\mathbb{Z}[\sqrt{2}] = \{a + b\sqrt{2} / a, b \in \mathbb{Z}\}$ is a Euclidean ring. **(06)**
b. If the degree of a polynomial $f(x) \in F[x]$ is n, then show that $f(x)$ has at most n distinct zeros in F. **(06)**
c. Let $f(x) \in F[x]$ be a polynomial of degree> 1. If $f(\alpha) = 0$ for some $\alpha \in F$, then show that $f(x)$ is reducible over F. **(02)**

OR

Q-2 **Attempt all questions** **(14)**

a. Determine all (a) quadratic (b) cubic and (c) biquadratic irreducible polynomials over $\mathbb{Z}_2$. **(06)**
b. Show that the integral domain $(\mathbb{Z}[i];\ +;\ \cdot)$ is a UFD. **(06)**
c. Let $\mathbb{Q}[\sqrt{-3}] = \left\{\frac{a+b\sqrt{-3}}{2} / a, b \in \mathbb{Z} \text{ and } a, b \text{ are both even or both odd}\right\}$ then show that the units of $\mathbb{Q}[\sqrt{-3}]$ are $\pm 1, \frac{\pm 1 \pm \sqrt{-3}}{2}$. **(02)**

Q-3 **Attempt all questions** **(14)**

a. State and prove Eisenstein criterion. **(07)**
b. Show that the commutative integral domain $\{a + b\sqrt{-5} / a, b \in \mathbb{Z}\}$ is not a UFD **(07)**

OR

Q-3 **a.** If p is a prime element of a UFD D and if $p/a_1a_2 \dots a_n$; $a_1, a_2 \dots a_n \in D$ then show that p/a_i, for some $i, 1 \leq i \leq n$. **(07)**

b. For nonzero polynomials show that, $f, g \in D[x], [fg] = [f] + [g]$. **(07)**

SECTION – II

Q-4 **Attempt the Following questions** **(07)**

a. Show that $x^2 + 1$ is irreducible over the integer mod 7. **(02)**

b. Show that $\sqrt{2} + \sqrt{3}$ is algebraic over $\mathbb{Q}$. **(02)**

c. Show that $\left[\mathbb{Q}(\sqrt{2}) : \mathbb{Q}\right] = 2$. **(02)**

d. Determine the characteristic of the ring $\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_6$. **(01)**

Q-5 **Attempt all questions** **(14)**

a. Let $F \subseteq E \subseteq K$ be fields. If $[K:E] < \infty$ and $[E:F] < \infty$, then show that **(07)**

(i) $[K:F] < \infty$,

(ii) $[K:F] = [K:E][E:F]$.

b. Let E be an extension field of F and let $u \in E$ algebraic over F. Let $p(x) \in F[x]$ be a polynomial of the least degree such that $p(u) = 0$. Then show that **(07)**

(i) $p(x)$ is irreducible over F,

(ii) If $g(x) \in F[x]$ is such that $g(u) = 0$, then $p(x)/g(x)$.

OR

Q-5 **Attempt all questions**

a. Let $f(x) \in F[x]$ be a nonconstant polynomial, then show that there exists an extension E of F in which $f(x)$ has a root. **(07)**

b. Define splitting field. Show that the degree of the extension field of $x^3 - 2$ over $\mathbb{Q}$ is 6. **(07)**

Q-6 **Attempt all questions** **(14)**

a. Show that $p(x) = x^2 - x - 1 \in \mathbb{Z}_3[x]$ is irreducible over $\mathbb{Z}_3$. Show that there exists an extension K of $\mathbb{Z}_3$ with nine elements having all roots of $p(x)$. **(06)**

b. Prove that a ring $\mathbb{Z}$ of all integers is an Euclidean ring. **(06)**

c. Examine the irreducibility of $f(x) = 2x^5 - 5x^4 + 5$ over $\mathbb{Q}$. **(02)**

OR

Q-6 **Attempt all questions**

a. If $\sqrt{3}$ and $\sqrt{5}$ both are algebraic over $\mathbb{Q}$ then find **(06)**

(i) degree of $\sqrt{3} + \sqrt{5}$ over $\mathbb{Q}$,

(ii) basis of $\mathbb{Q}(\sqrt{3}, \sqrt{5})$ over $\mathbb{Q}$.

b. Show that the splitting field of $f(x) = x^4 - 2 \in \mathbb{Q}[x]$ over $\mathbb{Q}$ is $\mathbb{Q}\left(2^{\frac{1}{4}}, i\right)$ and its degree of extension is 8. **(06)**

c. Determine the minimal polynomial of $\sqrt{2} - 3\sqrt{3}$ over $\mathbb{Q}$. **(02)**